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On integral invariants of ruled surface generated by the Darboux frame of the transversal intersection timelike curve of two timelike surfaces in Lorentz-Minkowski 3-space L^3

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In this paper, some characteristic properties of ruled surfaces which are generated by the Darboux frame of the transversal intersection timelike curve of two timelike surfaces were studied in Lorentz-Minkowski 3-Space L^3. Moreover, the relations between the Darboux frames, the Darboux derivate formulas, the apex angles, the pitchs, the geodesic curvatures, the normal curvatures, the geodesic torsions and the dralls of the ruled surfaces were given. Then, some characterizations were investigated for the transversal intersection curves. Namely, in the case of the intersection curves were geodesic lines and asymptotic lines, some corollaries were investigated. Lastly, examples were given and the figures were drawn using Maple.

Key words. Transversal intersection spaclike curve, ruled surface, geodesic curvatures.

INTRODUCTION

The surface-surface intersection (SSI) is a fundamental problem in computational geometry and geometric modelling of complex shapes. For general parametric surface intersections, the most commonly used methods include subdivision and marching. Marching-based algorithms begin by ending a starting point on a intersection curve, and proceed to march along the curve. Most marching methods make use of the local differential geometry or Taylor series expansions around each point of the intersection curve in order to give a direction and a control over each step in the procedure.

Two types of surfaces, parametric and implicit, are commonly used in geometric modelling systems. Those kinds of surfaces lead to three types of SSI problems: parametric-parametric, implicit-implicit and parametric-implicit. In general, what it is wanted in such problems is to determine the intersection curve between two given surfaces. To compute the intersection curve with precision

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and efficiency, approaches of superior order are necessary, that is, it is necessary to obtain the geometric properties of the intersection curves. While the differential geometry of a parametric curve can be found in many textbooks such as in Struik (1961) and Wilmore (1961); there is only a scarce literature on the differential geometry of intersection curves.

Willmore (1961) describes how to obtain the unit tangent vector t, the unit principal normal vector ***n***, and the unit binormal vector ***b***, as well as the curvature and the torsion of an intersection curve of two implicit surfaces. However, Ye and Maekawa (1999) provides t, n, b, κ, τ algorithms for the evaluation of higher-order derivatives for transversal as well as tangential intersections for all three types of intersection problems. Walrave (1985) studied the moving Frenet frames of curves in Minkowski space.

Aléssio (2006) introduced a method to compute the Frenet vector fields and the curvatures of the transversal intersection curves on implicit surfaces. Aléssio and Guadalupe (2007) studied the differential geometry of a transversal intersection spacelike curve resulting from the intersection of two parametric spacelike surfaces in Lorentz-Minkowski 3-space L^3. Also, Aléssio (2009), studied the intersection curve of three implicit surfaces in IR^4 by using implicit function theorem. However, for three parametric surfaces in E^4, the curvatures and the Frenet vectors of the intersection curve were given by Düldül (2010).

Çalişkan and Düldül (2010) studied the geodesic curvature and the geodesic torsion of the intersection curve for impilicit-implicit and parametric-parametric surfaces. Also, they gave the curvature and the curvature vector of intersection curve by using the normal vectors of surfaces. Sarioğlugil and Tutar (2007) studied the geodesic curvature and the fundamental forms of the regular surfaces in E^3.

In this paper, the relation between the Darboux frames of the transversal intersection timelike curve at the intersection point for two timelike surfaces was given in Lorentz space L^3. Also, the relations between the geodesic curvatures, the geodesic torsions and the normal curvatures were investigated. The apex angles, the pitches and the dralls were computed for the closed ruled surfaces generated by the Darboux frames and the relations between each other were showned in L^3.

PRELIMINARIES

Review of differential geometry in E^3

Here, we first will review some basic concept in E^3 for later use. Let $\alpha: I \to E^3$ be a differentiable curve with arc-length parameter s and $\{t, n, b\}$ be the Frenet frame of α at the point $\alpha(s)$, where

$$t(s) = \alpha'(s),\quad n(s) = \frac{\alpha''(s)}{\|\alpha''(s)\|},\quad b(s) = t(s) \wedge n(s)$$

The Frenet formulas of α are

$$\begin{bmatrix} t' \\ n' \\ b' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} t \\ n \\ b \end{bmatrix} \tag{1}$$

If α is a curve and x is a generator vector, then the ruled surface $X(s, v)$ has the following parameter representation:

$$X(s, v) = \alpha(s) + vx(s).$$

Namely, a ruled surface is a surface generated by the motion of a straight line x along α. Furthermore, if α is a closed curve, then this surfaces is called closed ruled surface. Moreover, the drall $P(x)$, the apex angle $\lambda(x)$ and the pitch $l(x)$ of the closed ruled surface are defined by:

$$P(x) = \frac{\det(\alpha', x, x')}{\|x\|^2},\quad \lambda(x) = \langle D, x\rangle,\quad l(x) = \langle V, x\rangle \tag{2}$$

respectively. Here, D and V are Steiner rotation vector and Steiner translation vector, respectively.

The Steiner translation vector V and Steiner rotation vector D are given as follows:

$$V = \oint_{(\alpha)} dx = t \oint_{(\alpha)} ds \tag{3}$$

$$D = \oint_{(\alpha)} w = t \oint_{(\alpha)} \tau ds + b \oint_{(\alpha)} \kappa ds, \tag{4}$$

Where

$$w = n \wedge n' = \tau t + \kappa b \tag{5}$$

is called Darboux vector (Figure 1).

If the Frenet vectors t, n, b are the straigth lines of the closed ruled surface, then we have

$$\begin{cases} \lambda_t = \oint_{(\alpha)} \tau ds \\ \lambda_n = 0 \\ \lambda_b = \oint_{(\alpha)} \kappa ds \end{cases},\quad \begin{cases} l_t = \oint_{(\alpha)} ds \\ l_n = 0 \\ l_b = 0 \end{cases},\quad \begin{cases} P_t = 0 \\ P_n = \frac{\tau}{\kappa^2+\tau^2}, \\ P_b = \frac{1}{\tau} \end{cases} \tag{6}$$

(Hacisalioğlu, 1983).

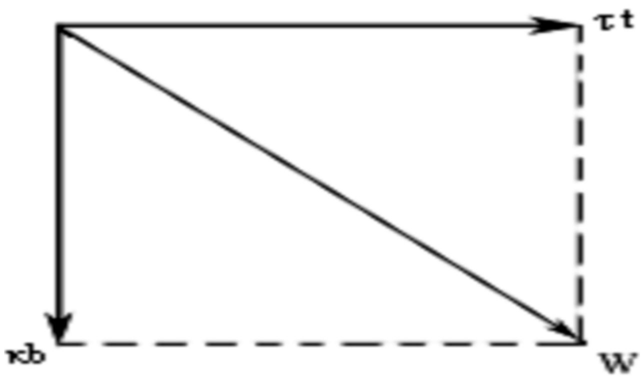

Figure 1. Darboux vector of a curve.

Definition 1

Let M be an oriented surface in E^3 and α be a unit speed curve on M. If t is the unit tangent vector of α, N is the unit normal vector of M and $g = N \wedge t$ the point $\alpha(s)$ of curve α, then $\{t, g, N\}$ is called the Darboux frame of α at that point. Thus, the Darboux formulas are:

$$\begin{bmatrix} t' \\ g' \\ N' \end{bmatrix} = \begin{bmatrix} 0 & \kappa_g & \kappa_n \\ -\kappa_g & 0 & \tau_g \\ -\kappa_n & -\tau_g & 0 \end{bmatrix} \begin{bmatrix} t \\ g \\ N \end{bmatrix} \qquad (7)$$

Where ψ is the angle between N and the unit principal normal n of α. Here $\kappa_n = \kappa cos\psi$, $\kappa_g = \kappa sin\psi$ and $\tau_g = \tau + \frac{d\psi}{ds}$ are called the normal curvature, the geodesic curvature and the geodesic torsion of α, respectively, (Kühnel, 1950).

Review of differential geometry in Lorentz-Minkowski 3-space L^3

Let $IR^3 = \{(x_1, x_2, x_3) \mid x_1, x_2, x_3 \in IR\}$ be a three-dimensional vector space, $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ be two vectors in IR^3, the Lorentz scalar product of x and y is defined by:

$$\langle x, y \rangle = -x_1 y_1 + x_2 y_2 + x_3 y_3$$

$(IR^3, \langle \ \rangle)$ is called three Lorentz space or Minkowski 3-space. We denote L^3 as $(IR^3, \langle \ \rangle)$. An arbitrary vector $x = (x_1, x_2, x_3) \in L^3$ can have one of three Lorentzian causal characters; it can be spacelike if $\langle x, x \rangle > 0$ or $x = \vec{0}$, timelike if $\langle x, x \rangle < 0$ and null (lightlike) if $\langle x, x \rangle = 0$. Similarly, an arbitrary curve $\alpha = \alpha(s)$. can locally be spacelike, timelike or null (lightlike), if all of its velocity vectors $\alpha'(s)$ are spacelike, timelike or null (lightlike), respectively. We say that a timelike vector is future pointing or past pointing if the first compound of the vector is positive or negative, respectively. For any vectors $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3) \in L^3$ in the meaning Lorentz vector product of x and y is defined by:

$$x \wedge y = \begin{vmatrix} e_1 & -e_2 & e_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = (x_2 y_3 - x_3 y_2, x_1 y_3 - x_3 y_1, x_2 y_1 - x_1 y_2)$$

Where

$$\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}, \quad e_1 \wedge e_2 = -e_3, \quad e_2 \wedge e_3 = e_1, \quad e_3 \wedge e_1 = -e_2$$

and

$$e_i = (\delta_{i1}, \delta_{i2}, \delta_{i3})$$

(Akutagava and Nishikawa, 1990).

Let $\{t, n, b\}$ denote the moving Frenet frame along the curve $\alpha(s)$ in the Lorentz space L^3. For an arbitrary timelike curve $\alpha(s)$ in L^3, the Frenet formulas are given as follows:

$$\begin{bmatrix} t' \\ n' \\ b' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ \kappa & 0 & -\tau \\ 0 & \tau & 0 \end{bmatrix} \begin{bmatrix} t \\ n \\ b \end{bmatrix} \qquad (8)$$

Where $\langle t, t \rangle = -1$, $\langle n, n \rangle = 1$, $\langle b, b \rangle = 1$ and $\langle t, n \rangle = \langle t, b \rangle = \langle n, b \rangle = 0$, (Uğurlu, 1996). Then, Darboux vector w of timelike curve α

$$w = \tau t - \kappa b. \qquad (9)$$

(Woestijne, 1990).

Definition 2

i) Hyperbolic angle: Let x and y be future pointing (or past pointing) timelike vectors in L^3. Then there is a unique real number $\theta \geq 0$ such that $\langle x, y \rangle = -\|x\|\|y\| cosh\theta$. This number is called the hyperbolic angle between the vectors x and y.

ii) Central angle: Let x and y be spacelike vectors in L^3 that span a timelike vector subspace. Then there is a unique real number $\theta \geq 0$ such that $\langle x, y\rangle = \|x\|\|y\|cosh\theta$. This number is called the central angle between the vectors x and y.

iii) Spacelike angle: Let x be spacelike vector and y be timelikelike vector in L^3 that span a spacelike vector subspace. Then there is a unique real number $\theta \geq 0$ such that $\langle x, y\rangle = \|x\|\|y\|cos\theta$. This number is called the spacelike angle between the vectors x and y.

iv) Lorentzian timelike angle: Let x and y be spacelike vectors in L^3. Then there is a unique real number $\theta \geq 0$ such that $\langle x, y\rangle = \|x\|\|y\|sinh\theta$. This number is called the Lorentzian timelike angle between the vectors x and y (Kazaz et al., 1883).

Definition 3

In the Lorentz 3-space, the following properties are satisfied:

(i) Two timelike vectors are never orthogonal.
(ii) Two null vectors are orthogonal if and only if they are linearly dependent.
(iii) A timelike vector is never orthogonal to a null (lightlike) vector (Kazaz et al., 1883).

Definition 4

A surface in the Lorentz 3-space L^3 is called a timelike surface if the induced metric on the surface is a Lorentz metric and is called a spacelike surface if the induced metric on the surface is a positive definite Riemannian metric, that is, the normal vector on the spacelike (timelike) surface is a timelike (spacelike) vector (Kazaz et al., 1883).

Let M be an oriented surface in three-dimensional Lorentz space L^3 and let consider a non-null curve $\alpha(s)$ lying on M fully. Since the curve $\alpha(s)$ is also in space, there exists Frenet frame $\{t, n, b\}$ at each points of the curve where t is unit tangent vector, n is principal normal vector and b is binormal vector, respectively.

Since the curve $\alpha(s)$ lies on the surface M, there exists another frame of the curve $\alpha(s)$ which is called Darboux frame and denoted by $\{t, g, N\}$. In this frame, t is the unit tangent of the curve, N is the unit normal of the surface M and g is a unit vector given by $g = N \wedge t$. Since the unit tangent t is common in both Frenet frame and Darboux frame, the vectors n, b, g and N lie on the same plane. Then, if the surface M is an oriented timelike surface, the curve $\alpha(s)$ lying on M is a timelike curve. So, the relations between the frames can be given as follows:

$$\begin{bmatrix} t \\ g \\ N \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -sin\psi & cos\psi \\ 0 & cos\psi & sin\psi \end{bmatrix} \begin{bmatrix} t \\ n \\ b \end{bmatrix} \quad (10)$$

Here, ψ is the angle between the spacelike vectors g and n (Kocayigit, 2004; Uğurlu and Topal, 1996; Uğurlu, 1997).

If the surface M is a timelike surface, then the curve $\alpha(s)$ lying on M is a timelike curve. Thus, the derivative formulae of the Darboux frame of $\alpha(s)$ is given by:

$$\begin{bmatrix} t' \\ g' \\ N' \end{bmatrix} = \begin{bmatrix} 0 & \kappa_g & \kappa_n \\ \kappa_g & 0 & -\tau_g \\ \kappa_n & \tau_g & 0 \end{bmatrix} \begin{bmatrix} t \\ g \\ N \end{bmatrix}, \quad \langle t, t\rangle = -1, \ \langle g, g\rangle = 1, \ \langle N, N\rangle = 1 \quad (11)$$

(Ugurlu and Kocayigit, 1996).

In the differential geometry of surfaces, for a curve $\alpha(s)$ lying on a surface M the followings are well-known

- $\alpha(s)$ is a geodesic curve $\Leftrightarrow$ $\kappa_g = 0$,
- $\alpha(s)$ is an asymptotic line $\Leftrightarrow$ $\kappa_n = 0$,
- $\alpha(s)$ is a principal line $\Leftrightarrow$ $\tau_g = 0$, (O'Neill, 1966).

PROBLEM STATEMENT

Let A and B be two regular timelike surfaces which have $X(u, v)$ and $Y(p, q)$ parametric representations, respectively, α be the transversal intersection curve of A and B with arc-length parameter, t be the unit tangent vector at the intersection point $P = X(u, v) = Y(p, q)$ and N^A and N^B be the unit normal vectors of A and B at the point P, respectively. Then, we have

$$N^A = \frac{X_u \wedge X_v}{\|X_u \wedge X_v\|}, \quad N^B = \frac{Y_p \wedge Y_q}{\|Y_p \wedge Y_q\|} \quad (12)$$

Since A and B intersect transversally, N^A and N^B is not parallel at the point P. Also, since the unit tangent vector

t of the intersection curve lies on the tangent planes of A and B

$$t = \frac{N^A \wedge N^B}{\| N^A \wedge N^B \|}. \tag{13}$$

Let $\{t, g^A, N^A\}$ and $\{t, g^B, N^B\}$ be Darboux frames of α at the point P where $g^A = N^A \wedge t$ and $g^B = N^B \wedge t$.

Since A and B be two timelike surfaces, the unit normal vectors N^A and N^B of surfaces A and B are spacelike vectors. Then, the unit vectors t is timelike vector and $g^A$and g^B are spacelike vectors. Also, since t is timelike vector, α is transversal intersection timelike curve. This case, the derivative formulas of the Darboux frames of $\alpha(s)$ are:

$$\begin{bmatrix} t' \\ (g^A)' \\ (N^A)' \end{bmatrix} = \begin{bmatrix} 0 & \kappa_g^A & \kappa_n^A \\ \kappa_g^A & 0 & -\tau_g^A \\ \kappa_n^A & \tau_g^A & 0 \end{bmatrix} \begin{bmatrix} t \\ g^A \\ N^A \end{bmatrix}, \tag{14}$$

$$\begin{bmatrix} t' \\ (g^B)' \\ (N^B)' \end{bmatrix} = \begin{bmatrix} 0 & \kappa_g^B & \kappa_n^B \\ \kappa_g^B & 0 & -\tau_g^B \\ \kappa_n^B & \tau_g^B & 0 \end{bmatrix} \begin{bmatrix} t \\ g^B \\ N^B \end{bmatrix}, \tag{15}$$

respectively. In these formulas, κ_g^A and κ_g^B, κ_n^A and κ_n^B, τ_g^A and τ_g^B are the geodesic curvatures, the normal curvatures and the geodesic torsions of α with respect to the surface A and B, respectively.

Theorem 1

Let α be the transversal intersection timelike curve of the timelike surfaces A and B with arc length parameter and let $\{t, g^A, N^A\}$ and $\{t, g^B, N^B\}$ be the Darboux frames of α at the point P, respectively. Then the relation between the Darboux frames of the timelike surfaces A and B is given as follows:

$$\begin{bmatrix} t \\ g^A \\ N^A \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} t \\ g^B \\ N^B \end{bmatrix}, \tag{16}$$

where θ is the angle between spacelike vectors N^A and N^B.

Proof

Let α be the transversal intersection timelike curve of the surfaces A and B. By using Equation 10, the Darboux frame $\{t, g^A, N^A\}$ and Frenet frame $\{t, n, b\}$ can be writen

$$\begin{bmatrix} t \\ g^A \\ N^A \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\sin\psi & \cos\psi \\ 0 & \cos\psi & \sin\psi \end{bmatrix} \begin{bmatrix} t \\ n \\ b \end{bmatrix} \tag{17}$$

Where ψ is the angle between spacelike vectors n and N^A. Similarly, by using Equation 10, the Darboux frame $\{t, g^B, N^B\}$ and Frenet frame $\{t, n, b\}$ can be written

$$\begin{bmatrix} t \\ n \\ b \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\sin\bar{\psi} & \cos\bar{\psi} \\ 0 & \cos\bar{\psi} & \sin\bar{\psi} \end{bmatrix} \begin{bmatrix} t \\ g^B \\ N^B \end{bmatrix} \tag{18}$$

Where $\bar{\psi}$ is the angle between spacelike vectors n and N^B. Substituting Equation 18 into Equation 17 we have

$$\begin{bmatrix} t \\ g^A \\ N^A \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\bar{\psi} - \psi) & \sin(\bar{\psi} - \psi) \\ 0 & -\sin(\bar{\psi} - \psi) & \cos(\bar{\psi} - \psi) \end{bmatrix} \begin{bmatrix} t \\ g^B \\ N^B \end{bmatrix} \tag{19}$$

or

$$\begin{bmatrix} t \\ g^A \\ N^A \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} t \\ g^B \\ N^B \end{bmatrix}$$

Where

$$\theta = \bar{\psi} - \psi. \tag{20}$$

Theorem 2

Let α be the closed transversal intersection timelike curve of the timelike surfaces A and B and let $\{t, g^A, N^A\}$ and $\{t, g^B, N^B\}$ be the Darboux frames of α at the point P, respectively. Then, the relations between the apex angles of ruled surfaces which are generated by the Darboux frames of α are given as follows:

$$\begin{cases} \lambda_t = -\oint_{(\alpha)} \tau ds \\ \lambda_{g^A} = (\cos\theta + \sin\theta \tan\bar{\psi})\lambda_{g^B}, \\ \lambda_{N^A} = (\cos\theta - \sin\theta \cot\bar{\psi})\lambda_{N^B} \end{cases} \tag{21}$$

Proof

Let ψ be the angle between spacelike vectors n and N^A and let $\bar{\psi}$ be the angle between spacelike vectors n and N^B. The apex angle of ruled surface which is generated by the timelike unit tangent vector t of the transversal intersection timelike curve α is

$$\lambda_t = \langle D, t \rangle = \langle \oint_{(\alpha)} w , t \rangle$$

Substituting Equation 9 into the last equation we obtain

$$\lambda_t = -\oint_{(\alpha)} \tau ds$$

The apex angle of ruled surface which is generated by the spacelike unit tangent vector b of the transversal intersection timelike curve α is

$$\lambda_b = \langle D, b \rangle = \langle \oint_{(\alpha)} w , b \rangle$$

Substituting Equation 9 into the last equation we obtain

$$\lambda_b = -\oint_{(\alpha)} \kappa ds. \tag{22}$$

The apex angle of ruled surface which is generated by the spacelike unit normal vector N^A of the timelike surface A is

$$\lambda_{N^A} = \langle D, N^A \rangle = \langle \oint_{(\alpha)} w , N^A \rangle$$

Substituting Equation 9 into the last equation we obtain

$$\lambda_{N^A} = -\sin\psi \oint_{(\alpha)} \kappa ds.$$

Substituting Equation 22 into the above equation we get

$$\lambda_{N^A} = \sin\psi \lambda_b. \tag{23}$$

From the last equation we have

$$\lambda_b = \frac{\lambda_{N^A}}{\sin\psi}. \tag{24}$$

Similarly, the apex angle of ruled surface which is generated by the spacelike unit normal vector N^B of the timelike surface B is

$$\lambda_b = \frac{\lambda_{N^B}}{\sin\bar{\psi}}. \tag{25}$$

From Equations 24 and 25 we get

$$\lambda_{N^A} = \frac{\sin\psi}{\sin\bar{\psi}} \lambda_{N^B}.$$

Substituting Equation 20 into the last equation we obtain

$$\lambda_{N^A} = (\cos\theta - \sin\theta \cot\bar{\psi})\lambda_{N^B}.$$

The apex angle of ruled surface which is generated by the spacelike unit vector g^A of the timelike surface A is

$$\lambda_{g^A} = \langle D, g^A \rangle = \langle \oint_{(\alpha)} w , g^A \rangle$$

Substituting Equation 9 into the last equation we obtain

$$\lambda_{g^A} = -\cos\psi \oint_{(\alpha)} \kappa ds.$$

Substituting Equation 22 into the above equation we get

$$\lambda_{g^A} = \cos\psi \lambda_b. \tag{26}$$

From the last equation we obtain

$$\lambda_b = \frac{\lambda_{g^A}}{\cos\psi}. \tag{27}$$

Similarly, the apex angle of ruled surface which is generated by the spacelike unit vector g^A of the timelike surface B is

$$\lambda_b = \frac{\lambda_{g^B}}{\cos\bar{\psi}}. \tag{28}$$

From Equations 27 and 28 we get

$$\lambda_{g^A} = \frac{\cos\psi}{\cos\bar{\psi}} \lambda_{g^B}$$

Subtituting Equation 20 into the last equation we obtain

$$\lambda_{g^A} = (\cos\theta + \sin\theta\tan\bar{\psi})\lambda_{g^B}$$

Theorem 3

Let α be the closed transversal intersection timelike curve of the timelike surfaces A and B and let $\{t, g^A, N^A\}$ and $\{t, g^B, N^B\}$ be the Darboux frames of α at the point P, respectively. Then, the relations between the pitches of ruled surfaces which are generated by the Darboux frames of α are given as follows:

$$l_t = \oint_{(\alpha)} ds, \quad l_{N^A} = l_{N^B} = l_{g^A} = l_{g^B} = 0. \tag{29}$$

Proof

The pitch of ruled surface which is generated by the timelike unit tangent vector of α is

$$l_t = \langle V, t \rangle. \tag{30}$$

Substituting Equation 3 into the last equation, we have

$$l_t = \oint_{(\alpha)} ds.$$

Similarly, the pitches of ruled surfaces which are generated by vectors N^A, N^B, g^A and g^B are

$$l_{N^A} = l_{N^B} = l_{g^A} = l_{g^B} = 0.$$

Theorem 4

Let α be transversal intersection timelike curve of timelike surfaces A and B and let $\{t, g^A, N^A\}$ and $\{t, g^B, N^B\}$ be the Darboux frame of α at the point P, respectively. Let τ_g^A be the geodesic torsion of α with respect to the timlike surface A and let τ_g^B be the geodesic torsion of α with respect to the timelike surface B. Then, the relation between the geodesic torsions is given as follows:

$$\tau_g^A = \tau_g^B - \frac{d\theta}{ds}. \tag{31}$$

Proof

From Equation 16 we know that

$$g^A = \cos\theta g^B + \sin\theta N^B. \tag{32}$$

Differentiating this equation, we obtain

$$(g^A)' = -\theta' \sin\theta g^B + \cos\theta (g^B)' + \theta' \cos\theta N^B + \sin\theta (N^B)'$$

From the Darboux formulas, we get

$$\kappa_g^A t - \tau_g^A N^A = (\cos\theta \kappa_g^A + \sin\theta \kappa_n^B)t + (-\theta' \sin\theta + \sin\theta \tau_g^B)g^B + (\theta' \cos\theta - \cos\theta \tau_g^B)N^B.$$

Multiplying the last equation with spacelike vector N^A, we have

$$\tau_g^A = \tau_g^B - \frac{d\theta}{ds}. \tag{34}$$

Theorem 5

Let α be transversal intersection timelike curve of timelike surfaces A and B and let $\{t, g^A, N^A\}$ and $\{t, g^B, N^B\}$ be the Darboux frame of α at the point P, respectively. Let κ_g^A and κ_n^A be the geodesic curvature and the normal curvature of timelike curve α with respect to the timelike surface A, respectively and let κ_g^B and κ_n^B be the geodesic curvature and the normal curvature of timelike curve α with respect to the timelike surface B, respectively. Then, the relations between the geodesic curvatures and the geodesic torsions are given as follows:

$$\begin{cases} \kappa_g^A = \cos\theta \kappa_g^B + \sin\theta \kappa_n^B \\ \kappa_n^A = -\sin\theta \kappa_g^B + \cos\theta \kappa_n^B{}' \end{cases} \tag{35}$$

Proof

From Equation 16 we know that

$$g^A = \cos\theta g^B + \sin\theta N^B \tag{36}$$

Differentiating this equation we obtain

$$(g^A)' = -\theta' \sin\theta g^B + \cos\theta (g^B)' + \theta' \cos\theta N^B + \sin\theta (N^B)'$$

From the Darboux formulas we get

$$\kappa_g^A t - \tau_g^A N^A = \left(\cos\theta\,\kappa_g^A + \sin\theta\,\kappa_n^B\right)t + \left(-\theta'\sin\theta + \sin\theta\,\tau_g^B\right)g^B + \left(\theta'\cos\theta - \cos\theta\,\tau_g^B\right)N^B. \quad (37)$$

Multiplying the last equation with timelike vector t, we have

$$\kappa_g^A = \cos\theta\,\kappa_g^B + \sin\theta\,\kappa_n^B \quad (38)$$

Also, from Equation 16 we know that

$$N^A = -\sin\theta\, g^B + \cos\theta\, N^B \quad (39)$$

Differentiating this equation we obtain

$$\kappa_n^A t + \tau_g^A g^A = \left(-\sin\theta\,\kappa_g^B + \cos\theta\,\kappa_n^B\right)t + \left(-\theta'\cos\theta + \cos\theta\,\tau_g^B\right)g^B + \left(-\theta'\sin\theta + \sin\theta\,\tau_g^B\right)N^B. \quad (40)$$

Multiplying the last equation with timelike vector t, we have

$$\kappa_n^A = -\sin\theta\,\kappa_g^B + \cos\theta\,\kappa_n^B \quad (41)$$

Thus, the following corollaries can be given.

Corollary 1: From Equations 38 and 41, it can be easily seen that

$$\left(\kappa_g^A\right)^2 + \left(\kappa_n^A\right)^2 = \left(\kappa_g^B\right)^2 + \left(\kappa_n^B\right)^2 \quad (42)$$

Corollary 2: Let α be transversal intersection timelike curve of timelike surfaces A and B. If α is the geodesic curve for the timelike surfaces A and B, then the relation between normal curvatures of α is

$$\kappa_n^A = \pm\kappa_n^B \quad (43)$$

Corollary 3: Let α be transversal intersection timelike curve of timelike surfaces A and B. If α is the asymptotic line for the timelike surfaces A and B, then the relation between geodesic curvatures of α is

$$\kappa_g^A = \pm\kappa_g^B \quad (44)$$

Theorem 6

Let α be transversal intersection timelike curve of timelike surfaces A and B and let $\{t, g^A, N^A\}$ and $\{t, g^B, N^B\}$ be the Darboux frame of α at the point P, respectively. If the angle between spacelike vectors N^A and N^B is constant along the curve α, then the relations between the dralls of ruled surfaces which are generated by the Darboux frames of α are given as follows:

$$\frac{1}{P_{g^A}} + \frac{1}{P_{N^A}} = \frac{1}{P_{g^B}} + \frac{1}{P_{N^B}} \quad (45)$$

Proof

The drall of ruled surface which is generated by the timelike unit tangent vector of α is

$$P_t = \frac{\det(\alpha', t, t')}{\|t'\|^2} = 0,$$

the drall of ruled surface which is generated by the spacelike unit normal vector N^A of the timelike surface A is

$$P_{N^A} = \frac{\det\left(\alpha', N^A, (N^A)'\right)}{\left\|(N^A)'\right\|^2} = \frac{-\tau_g^A}{\left(\kappa_n^A\right)^2 - \left(\tau_g^A\right)^2} \quad (46)$$

and the drall of ruled surface which is generated by the spacelike unit vector g^A of the timelike surface A is

$$P_{g^A} = \frac{\det\left(\alpha', g^A, (g^A)'\right)}{\left\|(g^A)'\right\|^2} = \frac{-\tau_g^A}{\left(\kappa_g^A\right)^2 - \left(\tau_g^A\right)^2} \quad (47)$$

Similarly, the dralls of ruled surfaces which are generated by N^B and g^B are:

$$P_{N^B} = \frac{-\tau_g^B}{\left(\kappa_n^B\right)^2 - \left(\tau_g^B\right)^2} \quad (48)$$

and

$$P_{g^B} = \frac{-\tau_g^B}{\left(\kappa_g^B\right)^2 - \left(\tau_g^B\right)^2} \quad (49)$$

From here, we get

$$\frac{1}{P_{g^A}} + \frac{1}{P_{N^A}} = \frac{-\left(\kappa_g^A\right)^2 - \left(\kappa_n^A\right)^2 + 2\left(\tau_g^A\right)^2}{\tau_g^A}$$

Substituting Equations 34 and 42 into the last equation we obtain

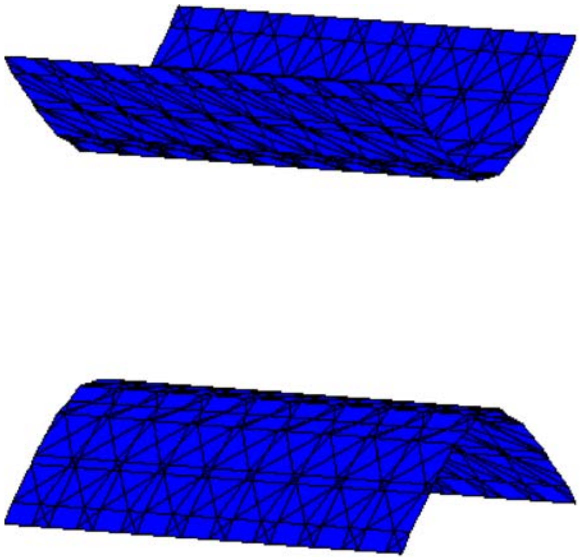

Figure 2. Timelike surface A.

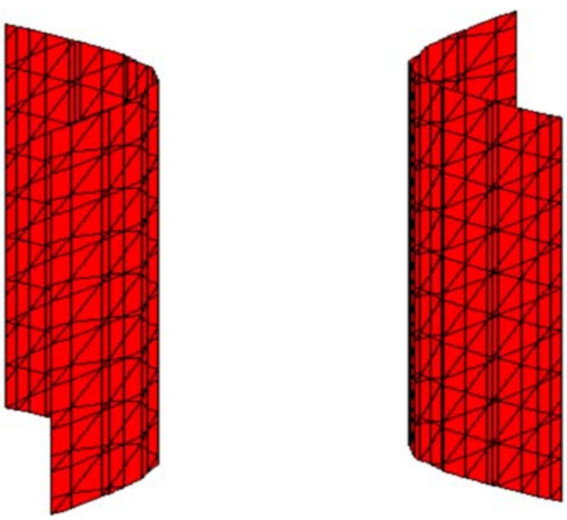

Figure 3. Timelike surface B.

$$\frac{1}{\rho_g{}^A}+\frac{1}{\rho_N{}^A}=\frac{-\left(\kappa_g^B\right)^2-\left(\kappa_n^B\right)^2+2\left(\tau_g^B-\frac{d\theta}{ds}\right)^2}{\tau_g^B-\frac{d\theta}{ds}}$$

Since θ is constant, it is seen that

$$\frac{1}{\rho_g{}^A}+\frac{1}{\rho_N{}^A}=\frac{1}{\rho_g{}^B}+\frac{1}{\rho_N{}^B} \quad (50)$$

Example 1: Let A and B be the timelike surfaces are given by Equation 45 $X(s,t)=(sinhs, t, coshs)$ and $Y(u,v)=(sinhu, coshu, v)$, respectively Figures 2 and 3. The transversal intersection timelike curves of timelike surfaces A and B are as follows as shown in Figures 4 and 5.

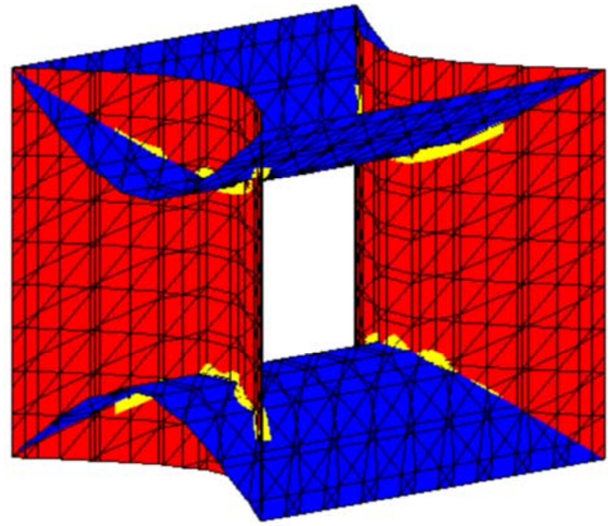

Figure 4. Intersection curves of timelike surfaces A and B.

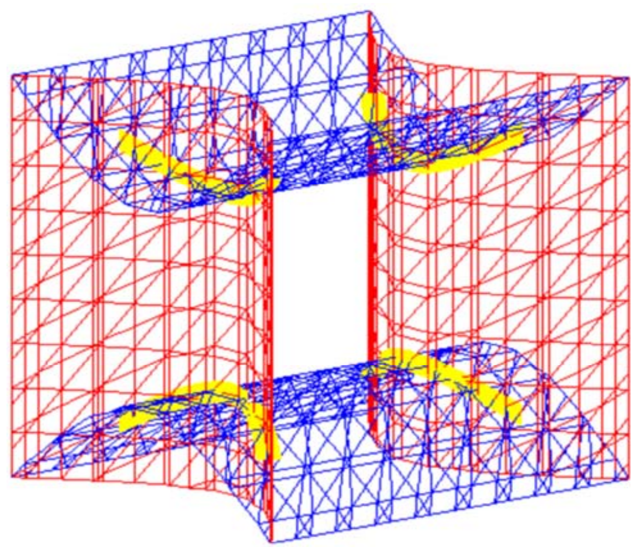

Figure 5. Intersection curves of timelike surfaces A and B.

Conflict of Interests

The author(s) have not declared any conflict of interests

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